Find $\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial y^2}$ in Polar Co-ords. So assume $u(x, y)$.

where $x = r\cos\theta$

$y = r\sin\theta$

$x^2 + y^2 = r^2$

Now $$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left[\frac{\partial u}{\partial r}\frac{\partial r}{\partial x}\right] + \frac{\partial}{\partial x}\left[\frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x}\right] = \frac{\partial}{\partial x}\left[\frac{\partial u}{\partial r}\right]\cdot\frac{\partial r}{\partial x} + \frac{\partial u}{\partial r}\frac{\partial^2 r}{\partial x} + \frac{\partial}{\partial x}\left[\frac{\partial u}{\partial \theta}\right]\frac{\partial \theta}{\partial x} + \frac{\partial u}{\partial \theta}\frac{\partial^2 \theta}{\partial x^2}$$

$$= \frac{\partial^2 u}{\partial r^2}\left[\frac{\partial r}{\partial x}\right]^2 + \frac{\partial^2 u}{\partial r\partial\theta}\frac{\partial r}{\partial x}\frac{\partial \theta}{\partial x} + \frac{\partial^2 u}{\partial r\partial\theta}\frac{\partial r}{\partial x}\frac{\partial \theta}{\partial x} + \frac{\partial^2 u}{\partial \theta^2}\left(\frac{\partial \theta}{\partial x}\right)^2 + \frac{\partial u}{\partial r}\frac{\partial^2 r}{\partial x^2} + \frac{\partial u}{\partial \theta}\frac{\partial^2 \theta}{\partial x^2}$$

$$= \frac{\partial^2 u}{\partial r^2}\left[\frac{\partial r}{\partial x}\right]^2 + 2\frac{\partial^2 u}{\partial r\partial\theta}\frac{\partial r}{\partial x}\frac{\partial \theta}{\partial x} + \frac{\partial^2 u}{\partial \theta^2}\left(\frac{\partial \theta}{\partial x}\right)^2 + \frac{\partial u}{\partial r}\left(\frac{\partial^2 r}{\partial x^2}\right) + \frac{\partial u}{\partial \theta}\left(\frac{\partial^2 \theta}{\partial x^2}\right)$$

similarly

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial r^2}\left[\frac{\partial r}{\partial y}\right]^2 + 2\frac{\partial^2 u}{\partial r\partial\theta}\frac{\partial r}{\partial y}\frac{\partial \theta}{\partial y} + \frac{\partial^2 u}{\partial \theta^2}\left(\frac{\partial \theta}{\partial y}\right)^2 + \frac{\partial u}{\partial r}\left(\frac{\partial^2 r}{\partial y^2}\right) + \frac{\partial u}{\partial \theta}\left(\frac{\partial^2 \theta}{\partial y^2}\right)$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial r^2}\left[\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2\right] + 2\frac{\partial^2 u}{\partial r\partial\theta}\left[r_x\theta_x + r_y\theta_y\right] + \frac{\partial u}{\partial \theta}\left[\theta_{xx} + \theta_{yy}\right]$$

$$+ \frac{\partial u}{\partial r}\left[r_{xx} + r_{yy}\right] + \frac{\partial^2 u}{\partial \theta^2}\left[\left(\frac{\partial \theta}{\partial x}\right)^2 + \left(\frac{\partial \theta}{\partial y}\right)^2\right]$$

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we find that

$$u_{xx} + u_{yy} = u_{rr} + \frac{u_{\theta\theta}}{r^2} + \frac{u_r}{r}$$

Find $\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial y^2}$ in Polar Co-ords. So assume $u(x,y)$.

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Now $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x}$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left[\frac{\partial u}{\partial r}\frac{\partial r}{\partial x}\right] + \frac{\partial}{\partial x}\left[\frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x}\right] = \frac{\partial}{\partial x}\left[\frac{\partial u}{\partial r}\right]\cdot\frac{\partial r}{\partial x} + \frac{\partial u}{\partial r}\frac{\partial^2 r}{\partial x^2} + \frac{\partial}{\partial x}\left[\frac{\partial u}{\partial \theta}\right]\frac{\partial \theta}{\partial x} + \frac{\partial u}{\partial \theta}\frac{\partial^2 \theta}{\partial x^2}$$

$$= \frac{\partial^2 u}{\partial r^2}\left[\frac{\partial r}{\partial x}\right]^2 + \frac{\partial^2 u}{\partial r\partial\theta}\frac{\partial r}{\partial x}\frac{\partial \theta}{\partial x} + \frac{\partial^2 u}{\partial r\partial\theta}\frac{\partial r}{\partial x}\frac{\partial \theta}{\partial x} + \frac{\partial^2 u}{\partial \theta^2}\left(\frac{\partial \theta}{\partial x}\right)^2 + \frac{\partial u}{\partial r}\frac{\partial^2 r}{\partial x^2} + \frac{\partial u}{\partial \theta}\frac{\partial^2 \theta}{\partial x^2}$$

$$= \frac{\partial^2 u}{\partial r^2}\left[\frac{\partial r}{\partial x}\right]^2 + 2\frac{\partial^2 u}{\partial r\partial\theta}\frac{\partial r}{\partial x}\frac{\partial \theta}{\partial x} + \frac{\partial^2 u}{\partial \theta^2}\left(\frac{\partial \theta}{\partial x}\right)^2 + \frac{\partial u}{\partial r}\left(\frac{\partial^2 r}{\partial x^2}\right) + \frac{\partial u}{\partial \theta}\left(\frac{\partial^2 \theta}{\partial x^2}\right)$$

similarly

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial r^2}\left[\frac{\partial r}{\partial y}\right]^2 + 2\frac{\partial^2 u}{\partial r\partial\theta}\frac{\partial r}{\partial y}\frac{\partial \theta}{\partial y} + \frac{\partial^2 u}{\partial \theta^2}\left(\frac{\partial \theta}{\partial y}\right)^2 + \frac{\partial u}{\partial r}\left(\frac{\partial^2 r}{\partial y^2}\right) + \frac{\partial u}{\partial \theta}\left(\frac{\partial^2 \theta}{\partial y^2}\right)$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial r^2}\left[\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2\right] + 2\frac{\partial^2 u}{\partial r\partial\theta}\left[r_x\theta_x + r_y\theta_y\right] + \frac{\partial u}{\partial r}\left[r_{xx} + r_{yy}\right] + \frac{\partial u}{\partial \theta}\left[\theta_{xx} + \theta_{yy}\right]$$

$$+ \frac{\partial^2 u}{\partial \theta^2}\left[\left(\frac{\partial \theta}{\partial x}\right)^2 + \left(\frac{\partial \theta}{\partial y}\right)^2\right]$$

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